



Exact 1/4 BPS loop—Chiral primary correlator

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Abstract

Correlation functions of 1/4 BPS Wilson loops with the infinite family of 1/2 BPS chiral primary operators are computed in $\mathcal{N} = 4$ super-Yang–Mills theory by summing planar ladder diagrams. Leading loop corrections to the sum are shown to vanish. The correlation functions are also computed in the strong-coupling limit by examining the supergravity dual of the loop–loop correlator. The strong coupling result is found to agree with the extrapolation of the planar ladders. The result is related to known correlators of 1/2 BPS Wilson loops and 1/2 BPS chiral primaries by a simple re-scaling of the coupling constant, similar to an observation of Drukker [N. Drukker, hep-th/0605151] for the case of the 1/4 BPS loop vacuum expectation value.

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Recently, the study of the properties of highly symmetric states has provided considerable insight into the AdS/CFT correspondence. In the case of 1/2 BPS local chiral operators and 1/2 BPS Wilson loops of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, their correspondence with 1/2 BPS gravitons and fundamental string world-sheets has been generalized to large operators where a beautiful picture of giant gravitons [1–3], giant Wilson loops [4–15] and bubbling geometries [16] has emerged. These relate infinite classes of highly symmetric protected operators in Yang–Mills theory to their dual geometries which solve IIB supergravity.

In the case of 1/2 BPS Wilson loops, an essential component of the bubbling loop picture is the ability to compute the loop expectation value and correlators of the loop with chiral primary operators in Yang–Mills theory by summing planar diagrams [11,17–21]. To point, for example, it is this sum, in the form of a matrix model computation, which provides evidence that the giant loops are dual to D3- and D5-branes. The matrix model is thought to coincide with the sum of all Feynman diagrams. This depends on cancellation of loop corrections, which has been demonstrated in leading orders, but has not yet been proven.³ It apparently holds for the expectation value of the 1/2 BPS Wilson loop and the correlator of the 1/2 BPS Wilson loop with any 1/2 BPS chiral primary operator. In all of these cases, when extrapolated to strong coupling, the sum of planar ladder Feynman diagrams agrees with the supergravity computation using AdS/CFT. This gives an infinite tower of functions which interpolate between weak and strong coupling. In this Letter, we will examine a modest extension of the picture. We will demonstrate similar results for the expectation value and the correlation functions of a 1/4 BPS Wilson loop with 1/2 BPS chiral operators.

The vacuum expectation value of the 1/4 BPS loop was studied by Drukker in Ref. [23]. He observed a number of interesting features of the gauge theory computation. One was that the ladder diagrams had a structure similar to the 1/2 BPS circle loop and they could be summed to obtain an expression very similar to the case of the 1/2 BPS loop. The difference was the replacement of the 't Hooft coupling λ by $\lambda \cos^2 \theta_0$ where θ_0 is a parameter of the 1/4 BPS loop. He further showed that, as occurred for the 1/2 BPS loop, the leading corrections from diagrams with internal vertices (those diagrams which are left out of the sum over

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³ There could also be non-perturbative contributions, which are plausibly suppressed in the large N limit [22].

ladders) cancel. He observed that, in the string dual where, following the prescription given in Ref. [24], the expectation value of the loop is found as the area of an extremal world-sheet bounding the loop, there are two saddle point solutions. He showed that the strong coupling extrapolation of the sum of diagrams on the gauge theory side carried a vestige of these two saddle points with some of the expected features of a saddle-point expansion.

In the following, we will study correlators of 1/4 BPS Wilson loops with 1/2 BPS chiral primary operators. We find that these correlators depend on the $SO(6)$ -orientation of the chiral primary. We identify all of the orientations where the Wilson loop and the chiral primary share some degree of supersymmetry. We find that the ladder diagrams can be summed for correlators of the loop and these operators and the result is identical to those previously found with the 1/2 BPS Wilson loop [11,20] with a certain rescaling of the coupling constant. We shall also study the strong coupling limit of the same correlators using the AdS/CFT correspondence. We identify the supergravity dual of the loop–loop correlation function and compute it in the asymptotic limit that is appropriate to extracting the contribution of intermediate chiral primary operators. This yields the limit of large N and large 't Hooft coupling λ . We find that the results agree with the extrapolation to strong coupling of the Yang–Mills computation.

The Wilson loop operator of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory which is most relevant to the AdS/CFT correspondence is [24]

$$W[C] = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left[\int_C (i A_\alpha(x(\tau)) \dot{x}^\alpha(\tau) + |\dot{x}(\tau)| \Theta^I(\tau) \Phi_I(x(\tau))) d\tau \right], \quad (1)$$

where $A_\alpha(x)$ are the gauge fields and $\Phi_I(x)$, $I = 1, \dots, 6$ are the scalar fields of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. The curve C is described by $x^\mu(\tau)$ and $\Theta^I(\tau)$, with $\sum_{I=1}^6 \Theta^I \Theta^I = 1$, describes a loop on the 5-sphere. This loop operator is related to the holonomy of heavy W -bosons in the gauge theory with $SU(N+1) \rightarrow SU(N) \times U(1)$ symmetry breaking. Its string theory dual is a source for a fundamental open string whose world-sheet ends on the contour C at the boundary of $AdS_5 \times S^5$.

When probed from a distance much larger than the extension of C , the Wilson loop operator should look like an assembly of local operators,⁴

$$W[C] = \langle 0 | W[C] | 0 \rangle \left(1 + \sum_{\Delta_i > 0} O_{\Delta_i}(0) L[C]^{\Delta_i} \xi_{\Delta_i}[C] \right), \quad (2)$$

where $L[C] = \int_C |\dot{x}(\tau)| d\tau$ is the length of C and we have assumed that C is near the origin 0. The operator expansion coefficients generally depend on the shape and orientation of C , as well as the parameters of $\mathcal{N} = 4$ Yang–Mills theory, the coupling constant g_{YM} and the number of colors N . In the remainder of this Letter, we will consider only the planar 't Hooft large N limit of Yang–Mills theory where $N \rightarrow \infty$ holding $\lambda \equiv g_{\text{YM}}^2 N$ fixed. In that limit, we can see from (4) below that ξ_Δ is the ratio of a disc to a cylinder amplitude and therefore should be of order $\frac{1}{N}$ times a function of λ .

All operators which can be made from the gauge fields, scalars and their derivatives can appear in the expansion in Eq. (2). We have classified operators according to their conformal dimensions, Δ_i . In a conformal field theory, the operators of fixed conformal dimensions can be organized into families which contain a primary operator with smallest Δ and an infinite tower of descendants. We will assume that primary operators are normalized so that

$$\langle 0 | O_\Delta(x) O_{\Delta'}(0) | 0 \rangle = \frac{\delta_{\Delta\Delta'}}{(4\pi^2 x^2)^\Delta}. \quad (3)$$

The operator expansion coefficient ξ_Δ for a primary operator can be extracted from the asymptotics of the correlator

$$\frac{\langle 0 | W[C] O_\Delta(x) | 0 \rangle}{\langle 0 | W[C] | 0 \rangle} = \frac{L[C]^\Delta}{(4\pi^2 |x|^2)^\Delta} \xi_\Delta + \dots \quad (4)$$

For example, for the 1/2 BPS circle Wilson loop,

$$C_{1/2}: \quad x^\mu(\tau) = (R \cos \tau, R \sin \tau, 0, 0), \quad \Theta^I = (1, 0, \dots) \quad (5)$$

a perturbative expansion of the loop gives

$$W[C_{1/2}] = \langle 0 | W[C_{1/2}] | 0 \rangle \left(\sum_{k=0}^{\infty} (2\pi R)^k \frac{1}{N k!} \frac{1}{2^k} \text{Tr}(Z(0) + \bar{Z}(0))^k : + \dots \right), \quad (6)$$

⁴ It is also possible to consider the insertion of supersymmetric operators into the Wilson loop itself. We emphasize that is a different procedure from what we are discussing here, where correlations of primary operators with the Wilson loop are the objects of most interest. Also, chiral operators of the type that we consider figure prominently in the discussion of the BMN limit as well as some issues of integrability [25–29].

where $Z = (\Phi_1 + i\Phi_2)$ and the dots indicate quantum corrections as well as operators with derivatives of Z , \bar{Z} and containing gauge fields. For the chiral primary operators

$$O_J \equiv \frac{1}{\sqrt{J\lambda^J}} : \text{Tr } Z(0)^J : \quad (7)$$

the weak coupling limit of $\xi_J[C_{1/2}, \lambda]$ is the appropriate coefficient in Eq. (6),

$$\xi_J[C_{1/2}; \lambda \sim 0] = \frac{1}{N} \frac{1}{2^J J!} \sqrt{J\lambda^J}. \quad (8)$$

This expression should receive quantum corrections. The sum of all quantum corrections from planar ladder diagrams was computed in Ref. [20]

$$\xi_J[C_{1/2}; \lambda] = \frac{1}{N} \frac{1}{2} \sqrt{\lambda^J} \frac{I_J(\sqrt{\lambda})}{I_1(\sqrt{\lambda})}, \quad (9)$$

where $I_J(x)$ is the J th modified Bessel function of the first kind. In the expression (9), as it must, the leading term in a small λ expansion agrees with (8). The leading order planar diagrams which are left out of the sum over ladders was also computed in Ref. [20] and were shown to cancel identically. It was then tempting to conjecture that these corrections vanish to all orders. To support this conjecture, the extrapolation of Eq. (9) to large λ can be compared with the result of a computation of the same coefficients using the AdS/CFT correspondence, originally done in Ref. [30],

$$\xi_J[C_{1/2}; \lambda \sim \infty] = \frac{1}{N} \frac{1}{2} \sqrt{\lambda^J}. \quad (10)$$

This coincides with the large λ limit of the expression in Eq. (9). The coefficients $\xi_J[C_{1/2}, \lambda]$ in (9), together with the result of Ref. [17]

$$\langle W[C_{1/2}] \rangle = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}), \quad (11)$$

yield an infinite family of interpolating functions which match both the strong and weak coupling limits computed in string and gauge theory, respectively.

In the present Letter, we will examine the 1/4 BPS loop which has the trajectory

$$C_{1/4}: \quad x^\mu(\tau) = R(\cos \tau, \sin \tau, 0, 0), \quad \Theta^I(\tau) = (\sin \theta_0 \cos \tau, \sin \theta_0 \sin \tau, \cos \theta_0, 0, 0, 0). \quad (12)$$

The main difference from the 1/2 BPS loop is that $\Theta^I(\tau)$ moves in a circle on an $S^2 \subset S^5$, rather than sitting at a point. Putting θ_0 to zero recovers the 1/2 BPS loop in (5). The special case of this 1/4 BPS loop with $\theta_0 = \pi/2$ was originally discussed by Zarembo [31].

To understand the supersymmetries of the loop with trajectory (12) we recall that the supersymmetry transformation of $\mathcal{N} = 4$ Yang–Mills theory is generated by the spinor

$$\epsilon(x) = \epsilon_0 + \gamma_\mu x^\mu \epsilon_1. \quad (13)$$

Here, we have to consider both Poincaré supersymmetries, with constant spinor ϵ_0 and conformal supersymmetries, with constant spinor ϵ_1 . In order to be supersymmetries of the 1/4 BPS Wilson loop, it is straightforward to see that they have to satisfy the equations [23]

$$\sin \theta_0 (\gamma^1 \Gamma^2 + \gamma^2 \Gamma^1) \epsilon_0 = 0, \quad \sin \theta_0 (\gamma^1 \Gamma^2 + \gamma^2 \Gamma^1) \epsilon_1 = 0, \quad (14)$$

$$\cos \theta_0 \epsilon_0 = R(-i\gamma^1 + \sin \theta_0 \Gamma^2) \Gamma^3 \gamma^2 \epsilon_1, \quad (15)$$

where the ten-dimensional gamma matrices are (γ^i, Γ^I) with $i = 1, \dots, 4$ and $I = 1, \dots, 6$. Let us count the supersymmetries. Each of the spinors ϵ_0 and ϵ_1 has 16 components. The conditions in (14) are half-rank and reduce the number of each of the spinors by half. Then (15) relates the remaining components of ϵ_1 to those of ϵ_0 in a way which is compatible with (14). The remaining independent components are eight—half of the original 16 components of ϵ_0 . This is 1/4 of the original 32 components of ϵ_0 and ϵ_1 .

We will consider a chiral operator which has an arbitrary $SO(6)$ orientation, beginning with

$$\text{Tr}(u \cdot \Phi(0))^J,$$

where u is a complex 6-vector, satisfying the constraint that $u^2 = 0$. Being a scalar operator, conformal supersymmetries are automatic. This operator has some Poincaré supersymmetry if there exist some non-zero constant spinors ϵ_0 which solve the

equation

$$u \cdot \Gamma \epsilon_0 = 0. \quad (16)$$

There are solutions only when $(u \cdot \Gamma)^2 = u^2 = 0$ which, as we have assumed, is the case. Then $u \cdot \Gamma$ is half-rank and there are exactly eight independent non-zero solutions of Eq. (16).

Now we can ask the question as to whether the eight independent ϵ_0 which solve (16) have anything in common with the eight solutions of (14) and (15), i.e. are there spinors which solve both of them?

Before we answer this question, let us backtrack to the case of the 1/2 BPS loop geometry (9). There Eq. (14) is absent and the spinors must solve (15) with $\theta_0 = 0$. This simply relates ϵ_1 to ϵ_0 , eliminating half of the possible spinors. There are 16 independent solutions of this equation—it is 1/2 BPS. Now, consider a chiral primary operator. Without loss of generality, we can consider the operator $\text{Tr}(\Phi_1 + i\Phi_2)^J$. It is supersymmetric if ϵ_0 satisfies the equation

$$(\Gamma^1 + i\Gamma^2)\epsilon_0 = 0.$$

The matrix $\Gamma_1 + i\Gamma_2$ has half-rank, so this requirement eliminates half of the supersymmetries generated by ϵ_0 . This leaves eight supersymmetries which commute with both the 1/2 BPS Wilson loop and the 1/2 BPS chiral primary operator. This high degree of residual joint supersymmetry is thought to be responsible for the fact that, apparently, only ladder diagrams contribute to the asymptotic limit of their correlator.

Returning to the 1/4 BPS loop and chiral primary with general orientation, it is easy to see that there is a simultaneous solution of (14)–(16) only when one of the following holds:

- $u_1 = u_2 = 0$. We can always do an $SO(6)$ rotation which commutes with the loop operator and sets $(u_4, u_5, u_6) \rightarrow (u_4, 0, 0)$. Then, there will be simultaneous solutions of (14)–(16) only when $u_3 = iu_4$ or when $u_3 = -iu_4$. In both of these cases, there are four solutions, corresponding to 1/8 supersymmetry in common between the chiral primary and the Wilson loop. Up to a constant, the chiral primary operator is $\text{Tr}(\Phi_3 + i\Phi_4)^J$ or the complex conjugate $\text{Tr}(\Phi_3 - i\Phi_4)^J$.
- $u_3 = u_4 = 0$. There is a solution when $u_1 = \pm iu_2$ and there is also 1/8 supersymmetry. The chiral primary is $\text{Tr}(\Phi_1 + i\Phi_2)^J$ or its complex conjugate. In this case, we show in Appendix C that the coefficient ξ_J which is extracted from the long range part of the correlator of this operator and the loop vanishes due to R-symmetry. Thus, for all $J > 0$, the coefficients of $\text{Tr}(\Phi_1 + i\Phi_2)^J$ or $\text{Tr}(\Phi_1 - i\Phi_2)^J$ in the operator expansion of the 1/4 BPS loop are zero.
- $u_1 = \pm iu_2$. There are two non-zero solutions when $u_3 = iu_4$ or when $u_3 = -iu_4$. This corresponds to 1/16 supersymmetry. There are essentially four operators,

$$\text{Tr}(\chi(\Phi_1 + i\Phi_2) + (\Phi_3 + i\Phi_4))^J$$

plus others with substitutions of $\Phi_1 - i\Phi_2$ or $\Phi_3 - i\Phi_4$. In this case too, because of R-symmetry the contribution with any non-zero power of $(\Phi_1 \pm i\Phi_2)$ will be zero. The coefficient $\xi_J[C_{1/4}]$ for these operators is therefore the same as those for the operator $\text{Tr}(\Phi_3 \pm i\Phi_4)^J$.

Thus we see that the interesting quantity where there is some degree of supersymmetry common to both the loop operator and the primary is

$$\xi_J[C_{1/4}] = \lim_{|x| \rightarrow \infty} \left(\frac{4\pi^2 |x|^2}{2\pi R} \right)^J \frac{1}{\sqrt{J\lambda^J}} \frac{\langle 0 | W[C_{1/4}] \text{Tr}(\Phi_3(x) + i\Phi_4(x))^J | 0 \rangle}{\langle 0 | W[C_{1/4}] | 0 \rangle}. \quad (17)$$

It is these partially supersymmetric configurations which we expect to have some level of protection from quantum corrections. Indeed, we shall find evidence for this. All other possibilities either vanish, are equivalent to (17) or have no supersymmetry at all. The cases with no supersymmetry at all are apparently not protected.

We will present arguments that the sum of planar ladder diagrams contributing to the correlation function in (17) gives a contribution which differs from the one for the 1/2 BPS loop quoted in Eq. (9) by the simple replacement $\lambda \rightarrow \lambda \cos^2 \theta_0$, so that the total result is

$$\xi_J[C_{1/4}] = \frac{1}{N} \frac{1}{2} \sqrt{\lambda \cos^2 \theta_0}^J \frac{I_J(\sqrt{\lambda \cos^2 \theta_0})}{I_1(\sqrt{\lambda \cos^2 \theta_0})}. \quad (18)$$

To find this result using Feynman diagrams, we begin with the lowest order diagrams, depicted in Fig. 1. There, each occurrence of the scalar Φ_3 in the composite operator contracts with a scalar Φ_3 in the Wilson loop. We consider only the planar diagrams. Each scalar Φ_3 from the Wilson loop carries a factor of $\cos \theta_0$, leading to an overall factor of $(\cos \theta_0)^J$. We are taking the convention for Feynman rules where each line in the Feynman diagram results in a factor of λ , totaling λ^J for the diagram in Fig. 1. With this convention, the chiral primary operator has normalization $\lambda^{-J/2}$ (see (7)). The net result is a factor of $\lambda^{J/2}$ which combines with

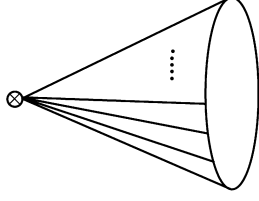


Fig. 1. The leading planar contribution to $\langle W[C_{1/4}] \text{Tr}(\Phi_3 + i\Phi_4)^J \rangle$. There are J lines connecting the chiral primary on the left with the circular Wilson loop on the right.

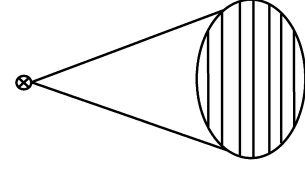


Fig. 2. A ladder diagram of $\langle W[C_{1/4}] \text{Tr}(\Phi_3 + i\Phi_4)^J \rangle$. The “rungs” represent the combined gauge field and scalar propagator. For clarity, J has been set to 2.

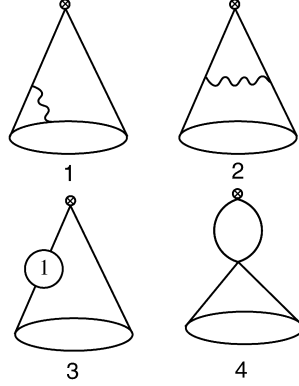


Fig. 3. The one-loop radiative corrections to $\langle W[C_{1/4}] \text{Tr}(\Phi_3 + i\Phi_4)^J \rangle$. Only an adjacent pair of the J scalar lines are shown.

the $(\cos \theta_0)^J$ to give a coupling constant dependence in the form $(\lambda \cos^2 \theta_0)^{J/2}$. This is identical to what one would have obtained by taking the same diagram for the 1/2 BPS loop and simply replacing λ by $\lambda \cos^2 \theta_0$.

To compute the next orders, we must decorate the diagram in Fig. 1 with propagators. The simplest are ladder diagrams, see Fig. 2, which go between two points on the periphery of the loop. They are described by summing the contribution of the vector and the scalar field. In the Feynman gauge, the sum of scalar and vector propagators connecting two points on arcs of the same circle is a constant:

$$\begin{aligned} & |\dot{x}(\sigma)| \Theta^I(\sigma) \langle \Phi_I(x(\sigma)) \Phi_J(x(\tau)) \rangle |\dot{x}(\tau)| \Theta^J(\tau) - \dot{x}^\alpha(\sigma) \langle A_\alpha(x(\tau)) A_\beta(x(\tau)) \rangle \dot{x}^\beta(\tau) \\ &= \frac{|\dot{x}(\sigma)| |\dot{x}(\tau)| \Theta(\sigma) \cdot \Theta(\tau) - \dot{x}(\sigma) \cdot \dot{x}(\tau)}{4\pi^2(x(\sigma) - x(\tau))^2} = \frac{R^2}{8\pi^2} \cos^2 \theta_0. \end{aligned}$$

This is what makes ladder diagrams easy to sum. We note that this propagator is accompanied by a factor of λ , so the total λ and θ_0 -dependence again comes in the combination $\lambda \cos^2 \theta_0$. Further, the only difference from the analogous quantity for the 1/2 BPS loop is the factor $\cos^2 \theta_0$. Thus we see that the sum of ladders for this 1/4 BPS loop will be identical to that for the 1/2 BPS loop with the replacement $\lambda \rightarrow \lambda \cos^2 \theta_0$.

Finally, there are the diagrams that have not yet been included so far. The conjecture is that they vanish. The leading order are depicted in Fig. 3. By a simple generalization of the argument obtained in Ref. [20] and explained in more detail in Ref. [26], they can be shown to cancel identically. Assuming that this cancellation occurs to higher orders as well, the result for the summation of all planar Feynman diagrams is summarized in the formula (18).

We now turn to the string theory dual of the correlator of the 1/4 BPS Wilson loop and the chiral primary operator. This will give a strong coupling planar limit of the operator expansion coefficients. It is most efficient to extract the operator expansion coefficient from the asymptotic form of the connected correlator of two Wilson loops, where the contributions of chiral primary intermediate states can be easily identified. This was used to compute the same quantity for a 1/2 BPS loop in Ref. [30]. The string theory dual of the Wilson loop operator is a fundamental string worldsheet which has as boundary the contour C and which itself sits at the boundary of the space $AdS_5 \times S^5$ [24]. The coupling constant of the string sigma model is $\alpha'/\mathcal{R}^2 = 1/\sqrt{\lambda}$ where \mathcal{R} is the radius of curvature of $AdS_5 \times S^5$ and we have used its relation with the 't Hooft coupling $\mathcal{R}^4/\alpha'^2 = \lambda$. In the limit of large λ , the worldsheet sigma model is weakly coupled and can be solved semi-classically. The leading order is classical, it simply finds an extremal surface with boundary C and which is compatible with other boundary conditions.

The connected loop-loop correlator has an extremal surface whose boundary is the two loops. When the loops have large separation, this surface degenerates to two disc geometry worldsheets whose boundaries are each loop with an infinitesimal tube connecting them, see figure Fig. 4. In the limit of large separation, this tube is described by the propagator of the lightest gravity

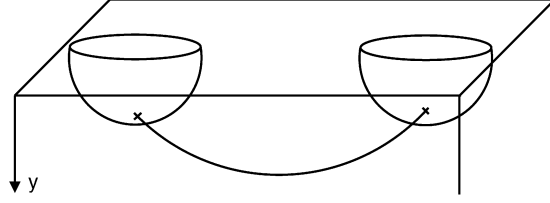


Fig. 4. The string worldsheets of two widely separated Wilson loops exchange a supergravity mode dual to a chiral primary operator.

modes, which at large λ are 1/2 BPS supergravitons, the string theory duals of the chiral primary operators. The connection between the graviton propagator and the worldsheet is through a vertex operator which must be identified and the connection point with the vertex operator must be integrated over the worldsheet. The resulting amplitude is proportional to the square of the desired operator expansion coefficient.

To begin, the first step is to identify the minimal surface in $AdS_5 \times S^5$ whose boundary is the 1/4 BPS contour $C_{1/4}$. This was done in Ref. [23]. We will summarize it here in more convenient coordinates. We take the metric of $AdS_5 \times S^5$

$$ds^2 = \sqrt{\lambda} \left(\frac{dy^2 + dr_1^2 + r_1^2 d\phi_1^2 + dr_2^2 + r_2^2 d\phi_2^2}{y^2} + d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta (d\rho^2 + \sin^2 \rho d\hat{\phi}^2 + \cos^2 \rho d\tilde{\phi}^2) \right). \quad (19)$$

The string world-sheet is then embedded as follows,

$$\begin{aligned} y &= R \tanh \sigma, & r_1 &= \frac{R}{\cosh \sigma}, & \phi_1 &= \tau, & r_2 &= 0, & \phi_2 &= \text{const}, \\ \sin \theta &= \frac{1}{\cosh(\sigma_0 \pm \sigma)}, & \phi &= \tau, & \rho &= \frac{\pi}{2}, & \hat{\phi} &= 0, & \tilde{\phi} &= \text{const}, \end{aligned} \quad (20)$$

where $\sigma \in [0, \infty]$ and $\tau \in [0, 2\pi]$ are the world-sheet coordinates. The contour $C_{1/4}$ is the boundary of the worldsheet at $\sigma = 0$, which in turn sits at $y = 0$, the boundary of $AdS_5 \times S^5$. The parameter $\cos \theta_0 = \frac{1}{\cosh \sigma_0}$. The choice of \pm sign in the embedding of θ arises because there are two saddle points in the classical action corresponding to wrapping the north or south pole of the S^5 . Of course the sign should be chosen to minimize the classical action, which corresponds to choosing $+$. The other saddle point is unstable, and the string world-sheet will slip-off the unstable pole.

The supergravity modes that we are interested in are fluctuations of the RR 5-form as well as the spacetime metric. They are by now very well known, and details can be found in Refs. [15,30,32–34]. The fluctuations are

$$\delta g_{\alpha\beta} = \left[-\frac{6J}{5} g_{\alpha\beta} + \frac{4}{J+1} D_{(\alpha} D_{\beta)} \right] s^J(X) Y_J(\Omega), \quad \delta g_{IK} = 2k g_{IK} s^J(X) Y_J(\Omega), \quad (21)$$

where α, β are AdS_5 and I, K are S^5 indices. The symbol X indicates coordinates on AdS^5 and Ω coordinates on the S^5 . The $D_{(\alpha} D_{\beta)}$ represents the traceless symmetric double covariant derivative. The $Y_J(\Omega)$ are the spherical harmonics on the five-sphere, while $s^J(X)$ have arbitrary profile and represent a scalar field propagating on AdS_5 space with mass squared $= J(J-4)$, where J labels the representation of $SO(6)$ and must be an integer greater than or equal to 2. (This is the representation of $SO(6)$ which contains the chiral primary operators that we are interested in.)

The supergravity field dual to the operator $\text{Tr}(u \cdot \Phi)^J$ is obtained by choosing the combination of spherical harmonics with the same quantum numbers and evaluating them on the worldsheet using (20) (see Appendix B) so that,

$$Y_J(\theta, \phi) = \mathcal{N}_J(u) [u_1 \sin \theta \cos \phi + u_2 \sin \theta \sin \phi + u_3 \cos \theta]^J. \quad (22)$$

The worldsheets will be connected by the propagator for the scalar supergravity mode $s^J(X)$. The asymptotic form of this propagator for large separation x is

$$P(X, \bar{X}) = \langle s^J(X) s^J(\bar{X}) \rangle \simeq \Lambda_J \left(\frac{1}{x} \right)^{2J} y^J \bar{y}^J, \quad (23)$$

where $\Lambda_J = 2^J (J+1)^2 / (16N^2 J)$. The barred quantities are coordinates on the second Wilson loop worldsheet. Then, in the large λ limit, the Wilson loop correlator is

$$\frac{\langle 0 | W[C_{1/4}, x] W^*[C_{1/4}, 0] | 0 \rangle}{|\langle 0 | W[C_{1/4}] | 0 \rangle|^2} = \iint_{\Sigma \times \bar{\Sigma}} \partial_a X^M \partial^a X^N \delta g_{MN} P(X, \bar{X}) \delta \bar{g}_{\bar{M}\bar{N}} \partial_{\bar{a}} X^{\bar{M}} \partial^{\bar{a}} X^{\bar{N}}, \quad (24)$$

where $M, N = 1, \dots, 10$ and the δg_{MN} are given in (21), except now we have removed the fluctuating parts, $s^J(X)$ and replaced them by the propagator P . The pullback of the fluctuations (21) to the worldsheet are found in Appendix A. Using them we have,

$$\frac{\langle 0|W[C_{1/4}, x]W^*[C_{1/4}, 0]|0\rangle}{|\langle 0|W[C_{1/4}]|0\rangle|^2} = \frac{\Lambda_J}{x^{2J}} \frac{\lambda}{16\pi^2} \left[2J \int d\sigma d\tau y'^2 y^{J-2} Y_J(\theta, \phi) - 2J \int d\sigma d\tau (r_1'^2 + r_1^2) y^{J-2} Y_J(\theta, \phi) \right. \\ \left. + 2J \int d\sigma d\tau (\theta'^2 + \sin^2 \theta) y^J Y_J(\theta, \phi) \right]^2. \quad (25)$$

Each of the terms inside the square on the right-hand side of the above expression has a common factor of

$$\int_0^{2\pi} d\tau Y_J(\theta, \phi) = \mathcal{N}_J(u) \int_0^{2\pi} d\tau [u_1 \sin \theta \cos \tau + u_2 \sin \theta \sin \tau + u_3 \cos \theta]^J. \quad (26)$$

From this expression we see that, consistent with our expectations using R-symmetry on the gauge theory side, for the at least 1/16 supersymmetric combination of loop and primary when $u_2 = \pm i u_1$, the dependence on u_1 and u_2 integrates to zero. If these parameters are chosen more arbitrarily, so that there is no supersymmetry at all, the loop depends on them. In that case the contributions proportional to powers of u_1 and u_2 in the final result for the operator expansion coefficients do not follow the rule that they are related to the 1/2 BPS loop ones by the replacement of λ by $\lambda \cos^2 \theta_0$. We attribute this to absence of supersymmetry. From here, we will proceed with the supersymmetric case only by putting $u_1 = u_2 = 0$ and $u_3 = 1$.

We will now compute the integrals in (25) with this assumption. We note that the embedding (20) has some nice properties. For instance $y'^2 + r_1'^2 = r_1^2 = y'^2$ and also $\sin^2 \theta = \theta'^2$. Using these, we can express the integrals in (25) as follows,

$$\frac{2^{-J/2}}{R^J} \int d\sigma y'^2 y^{J-2} \cos^J \theta = 2^{-J/2} \int_0^\infty d\sigma \frac{(\tanh \sigma)^{J-2}}{\cosh^4 \sigma} \tanh^J(\sigma_0 \pm \sigma) = 2^{-J/2} \int_0^1 dz (1 - z^2) z^{J-2} \left(\frac{\pm z + \cos \theta_0}{1 \pm z \cos \theta_0} \right)^J, \quad (27)$$

$$\frac{2^{-J/2}}{R^J} \int d\sigma (r_1'^2 + r_1^2) y^{J-2} \cos^J \theta = 2^{-J/2} \int_0^1 dz (1 + z^2) z^{J-2} \left(\frac{\pm z + \cos \theta_0}{1 \pm z \cos \theta_0} \right)^J, \quad (28)$$

$$\frac{2^{-J/2}}{R^J} \int d\sigma (\theta'^2 + \sin^2 \theta) y^J \cos^J \theta = -2^{1-J/2} \int_{\mp \cos \theta_0}^{-1} dz \left(\frac{\pm z + \cos \theta_0}{1 \pm z \cos \theta_0} \right)^J z^J. \quad (29)$$

Putting everything together,

$$\frac{\langle 0|W[C_{1/4}, x]W^*[C_{1/4}, 0]|0\rangle}{|\langle 0|W[C_{1/4}]|0\rangle|^2} = 16J^2 \frac{\Lambda_J}{2^J} \left(\frac{R}{x} \right)^{2J} \frac{\lambda}{4} \left[\left\{ \int_{-1}^{\mp \cos \theta_0} dz - \int_0^1 dz \right\} \left(\frac{\pm z + \cos \theta_0}{1 \pm z \cos \theta_0} \right)^J z^J \right]^2 \\ = 16J^2 \frac{\Lambda_J}{2^J} \left(\frac{R}{x} \right)^{2J} \frac{\lambda}{4} \left[\frac{-(\pm)^{J+1} \cos \theta_0}{J+1} \right]^2 = \frac{1}{4N^2} J \lambda \cos^2 \theta_0 \left(\frac{R}{x} \right)^{2J}, \quad (30)$$

which is just the result for the 1/2 BPS circle [30] with $\lambda \rightarrow \lambda \cos^2 \theta_0$. Using the prescription [30] to obtain from the loop-to-loop correlator the overlap with the chiral primary in question, we find $\xi_J[C_{1/4}] = \sqrt{J \lambda \cos^2 \theta_0} / 2N$. This is identical to the large λ limit of Eq. (18). We have thus confirmed that the sum of planar ladder diagrams agrees with the prediction of AdS/CFT in the strong coupling limit. The emergence of this structure on the supergravity side of the duality is non-trivial. The integrations over the AdS_5 and S^5 portions of the string worldsheet conspire in a complicated way in (30) to give the $\lambda \rightarrow \cos^2 \theta_0 \lambda$ result.

It is instructive to consider this calculation where both saddle points of the classical action are kept in the path integral, as is discussed in [23]. There it was noted that the semi-classical result for the expectation value of the Wilson loop is a sum of two terms; one proportional to $\exp(\sqrt{\lambda'})$ and the other to $\exp(-\sqrt{\lambda'})$, where $\lambda' = \cos^2 \theta_0 \lambda$. This was mirrored in the asymptotic expansion [35] of the modified Bessel function of (11),

$$I_1(\sqrt{\lambda'}) = \frac{e^{\sqrt{\lambda'}}}{\sqrt{2\pi\sqrt{\lambda'}}} \sum_{k=0}^{\infty} \left(\frac{-1}{2\sqrt{\lambda'}} \right)^k \frac{\Gamma(3/2+k)}{k! \Gamma(3/2-k)} \pm i \frac{e^{-\sqrt{\lambda'}}}{\sqrt{2\pi\sqrt{\lambda'}}} \sum_{k=0}^{\infty} \left(\frac{1}{2\sqrt{\lambda'}} \right)^k \frac{\Gamma(3/2+k)}{k! \Gamma(3/2-k)}, \quad (31)$$

where the sign of the i is ambiguous due to the Stokes' Phenomenon [36]. The factor of i was associated with the fluctuation determinant of the three tachyonic modes associated with the worldsheet slipping off the unstable pole of the five-sphere.

Due to the sign structure found in (30) before squaring, the analogous structure for the connected correlator of the primary with the loop is a sum of a term proportional to $\exp(\sqrt{\lambda'})$ and of another proportional to $(-1)^{J+1} \exp(-\sqrt{\lambda'})$. The sum of these

two terms should then be normalized by the expectation value of the Wilson loop. If we employ the asymptotic expansions of the modified Bessel functions in (9), we have

$$\frac{I_J(\sqrt{\lambda'})}{I_1(\sqrt{\lambda'})} = \frac{e^{\sqrt{\lambda'}} \sum_{k=0}^{\infty} \left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(J+k+1/2)}{k! \Gamma(J-k+1/2)} \mp i(-1)^J e^{-\sqrt{\lambda'}} \sum_{k=0}^{\infty} \left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(J+k+1/2)}{k! \Gamma(J-k+1/2)}}{e^{\sqrt{\lambda'}} \sum_{k=0}^{\infty} \left(\frac{-1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k! \Gamma(3/2-k)} \pm i e^{-\sqrt{\lambda'}} \sum_{k=0}^{\infty} \left(\frac{1}{2\sqrt{\lambda'}}\right)^k \frac{\Gamma(3/2+k)}{k! \Gamma(3/2-k)}}. \quad (32)$$

This clearly reflects the presence of two saddle points in the functional integrals in both the numerator and denominator.

We also note that the chiral primary has zero overlap with the supersymmetric Wilson loop (i.e. $W_{\theta_0=\pi/2}$). This is expected, since two such Wilson loops should not interact with each other by supersymmetry.

There has been extensive work of late concerning Wilson loops whose $SU(N)$ representations are of higher rank [4–8,10–13]. They have been associated with D-brane solutions analogous to giant gravitons. Explicit solutions are available for the 1/2 BPS loop, and results have been matched to matrix model calculations. It would be very interesting to solve the DBI equations of motion corresponding to the 1/4 BPS loop, and to repeat the calculations done here for that solution, as has been recently done for the 1/2 BPS case [15].

Appendix A. Metric fluctuations

Given (21) and (19), we must construct the traceless symmetric double covariant derivative,

$$D_{(\mu} D_{\nu)} \equiv \frac{1}{2}(D_{\mu} D_{\nu} + D_{\nu} D_{\mu}) - \frac{1}{5} g_{\mu\nu} g^{\rho\sigma} D_{\rho\sigma}. \quad (A.1)$$

The action of $D_{\mu} D_{\nu}$ on a scalar field ϕ is,

$$D_{\mu} D_{\nu} \phi = \partial_{\mu} \partial_{\nu} \phi - \Gamma_{\mu\nu}^{\lambda} \partial_{\lambda} \phi. \quad (A.2)$$

The Christoffel symbols for the AdS geometry (19) are,

$$\Gamma_{\phi_i \phi_i}^{r_i} = -r_i, \quad \Gamma_{\phi_i \phi_i}^y = \frac{r_i^2}{y}, \quad \Gamma_{\phi_i r_i}^{\phi_i} = \frac{1}{r_i}, \quad \Gamma_{\phi_i y}^{\phi_i} = -\frac{1}{y}, \quad \Gamma_{r_i r_i}^y = \frac{1}{y}, \quad \Gamma_{y r_i}^{r_i} = -\frac{1}{y}, \quad \Gamma_{yy}^y = -\frac{1}{y}, \quad (A.3)$$

where $i = 1, 2$. The trace of $D_{\mu} D_{\nu} \phi$ is given by,

$$g^{\mu\nu} D_{\mu} D_{\nu} \phi = \sum_{i=1}^2 \left(y^2 \partial_y^2 + y^2 \partial_{r_i}^2 + \frac{y^2}{r_i^2} \partial_{\phi_i}^2 - 3y \partial_y + \frac{y^2}{r_i} \partial_{r_i} \right) \phi. \quad (A.4)$$

Because of (23), we only keep those terms of $D_{(\mu} D_{\nu)}$ which contain derivatives in y . These are,

$$D_{(y} D_{y)} = \frac{4}{5} \partial_y^2 + \frac{8}{5y} \partial_y, \quad D_{(r_1} D_{r_1)} = \frac{1}{r_1^2} D_{(\phi_1} D_{\phi_1)} = -\frac{1}{5} \partial_y^2 - \frac{2}{5y} \partial_y. \quad (A.5)$$

We now note that since the derivatives will be acting on y^J from the propagator, we may replace $\partial_y^2 \rightarrow J(J-1)/y^2$ and $y^{-1} \partial_y \rightarrow J/y^2$. Therefore the metric fluctuations may be expressed as follows,

$$\begin{aligned} \delta g_{yy} &= \left[-\frac{6J}{5} + \frac{4}{J+1} \left(\frac{4}{5} J(J-1) + \frac{8}{5} J \right) \right] \frac{L^2}{y^2} = 2J \frac{L^2}{y^2}, \\ \delta g_{r_1 r_1} &= \frac{1}{r_1^2} \delta g_{\phi_1 \phi_1} = \left[-\frac{6J}{5} - \frac{4}{J+1} \left(\frac{1}{5} J(J-1) + \frac{2}{5} J \right) \right] \frac{L^2}{y^2} = -2J \frac{L^2}{y^2}. \end{aligned} \quad (A.6)$$

Appendix B. Spherical harmonics

The five-sphere is embedded in \mathbb{R}^6 in the following manner,

$$\begin{aligned} x^1 &= \sin \theta \cos \phi, & x^2 &= \sin \theta \sin \phi, \\ x^3 &= \cos \theta \sin \rho \cos \hat{\phi}, & x^4 &= \cos \theta \sin \rho \sin \hat{\phi}, \\ x^5 &= \cos \theta \cos \rho \cos \tilde{\phi}, & x^6 &= \cos \theta \cos \rho \sin \tilde{\phi}, \end{aligned} \quad (B.1)$$

and has the metric

$$ds_{S^5}^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta (d\rho^2 + \sin^2 \rho d\hat{\phi}^2 + \cos^2 \rho d\tilde{\phi}^2). \quad (B.2)$$

The embedding (20) takes $\rho = \pi/2$, $\hat{\phi} = 0$, or $x^4 = x^5 = x^6 = 0$. Note that $\rho \in [0, \pi/2]$ while $\theta \in [0, \pi]$. A general chiral primary normalized as in (3) may be written as,

$$\frac{2^{J/2}}{\sqrt{\lambda^J J}} C^{I_1 \dots I_J} \text{Tr} \Phi_{I_1} \dots \Phi_{I_J}, \quad (\text{B.3})$$

where $C^{I_1 \dots I_J}$ is traceless symmetric and $C^{I_1 \dots I_J} C^{*I_1 \dots I_J} = 1$. The corresponding spherical harmonic is given by $Y_J(\theta, \phi) = C^{I_1 \dots I_J} x^{I_1} \dots x^{I_J}$. A properly normalized (i.e. (3)) operator built on $\text{Tr}(u \cdot \Phi)^J$ will then correspond to

$$Y_J(\theta, \phi) = \mathcal{N}_J(u) [u_1 \sin \theta \cos \phi + u_2 \sin \theta \sin \phi + u_3 \cos \theta]^J \quad (\text{B.4})$$

for some normalization $\mathcal{N}_J(u)$. If we choose $u_1 = u_2 = 0$ and $u_3 = \pm i u_4 = 1$, i.e. the operator $\text{Tr}(\Phi_3 \pm i \Phi_4)^J / \sqrt{\lambda^J J}$, then $\mathcal{N}_J(u) = 2^{-J/2}$.

Appendix C. R-symmetry

Let $\mathcal{O}_J = \frac{1}{\sqrt{J\lambda}} \text{Tr}(\Phi_1 + i\Phi_2)^J$ and let U be a rotation in the x^1 – x^2 plane. Then

$$\langle \mathcal{O}_J(x) W[C_{1/4}] \rangle = \langle U \mathcal{O}_J(x) W[C_{1/4}] U^\dagger \rangle = \langle \mathcal{O}_J(Ux) U W[C_{1/4}] U^\dagger \rangle. \quad (\text{C.1})$$

Examining $C_{1/4}$ in (12), we see that the spatial rotation acting on $W[C_{1/4}]$ may be realized by a shift in the contour parameter τ , which can in turn be compensated by an R-symmetry rotation R in the Θ^1 – Θ^2 plane, $U W[C_{1/4}] U^\dagger = R W[C_{1/4}] R^\dagger$. Then,

$$\langle \mathcal{O}_J(x) W[C_{1/4}] \rangle = \langle R \mathcal{O}_J(Ux) R^\dagger W[C_{1/4}] \rangle. \quad (\text{C.2})$$

The operator expansion coefficient depends on the leading asymptotic in large x which is a function of only the length of $C_{1/4}$ and x^2 ,

$$\langle \mathcal{O}_J(x) W[C_{1/4}] \rangle \simeq \left(\frac{2\pi R}{4\pi^2 x^2} \right)^J \xi_J + \dots \quad (\text{C.3})$$

Performing the Θ^1 – Θ^2 plane R-symmetry transformation on \mathcal{O}_J multiplies it by a phase $\exp(iJ\phi)$ so that,

$$\langle R \mathcal{O}_J(Ux) R^\dagger W[C_{1/4}] \rangle \simeq e^{iJ\phi} \left(\frac{2\pi R}{4\pi^2 (Ux)^2} \right)^J \xi_J + \dots = e^{iJ\phi} \left(\frac{2\pi R}{4\pi^2 x^2} \right)^J \xi_J + \dots \quad (\text{C.4})$$

Using (C.2) and (C.3), we have $e^{iJ\phi} \xi_J = \xi_J$, i.e. $\xi_J = 0$.

References

- [1] J. McGreevy, L. Susskind, N. Toumbas, JHEP 0006 (2000) 008, hep-th/0003075.
- [2] V. Balasubramanian, M. Berkooz, A. Naqvi, M.J. Strassler, JHEP 0204 (2002) 034, hep-th/0107119.
- [3] S. Corley, A. Jevicki, S. Ramgoolam, Adv. Theor. Math. Phys. 5 (2002) 809, hep-th/0111222.
- [4] N. Drukker, B. Fiol, JHEP 0502 (2005) 010, hep-th/0501109.
- [5] S. Yamaguchi, hep-th/0601089.
- [6] S. Yamaguchi, JHEP 0605 (2006) 037, hep-th/0603208.
- [7] J. Gomis, F. Passerini, hep-th/0604007.
- [8] D. Rodríguez-Gomez, Nucl. Phys. B 752 (2006) 316, hep-th/0604031.
- [9] A. Dymarsky, S. Gubser, Z. Guralnik, J.M. Maldacena, hep-th/0604058.
- [10] O. Lunin, JHEP 0606 (2006) 026, hep-th/0604133.
- [11] K. Okuyama, G.W. Semenoff, JHEP 0606 (2006) 057, hep-th/0604209.
- [12] S.A. Hartnoll, S.P. Kumar, hep-th/0605027.
- [13] S.A. Hartnoll, hep-th/0606178.
- [14] B. Chen, W. He, hep-th/0607024.
- [15] S. Giombi, R. Ricci, D. Trancanelli, hep-th/0608077.
- [16] H. Lin, O. Lunin, J.M. Maldacena, JHEP 0410 (2004) 025, hep-th/0409174.
- [17] J.K. Erickson, G.W. Semenoff, K. Zarembo, Nucl. Phys. B 582 (2000) 155, hep-th/0003055.
- [18] N. Drukker, D.J. Gross, J. Math. Phys. 42 (2001) 2896, hep-th/0010274.
- [19] G. Akemann, P.H. Damgaard, Phys. Lett. B 513 (2001) 179, hep-th/0101225;
- G. Akemann, P.H. Damgaard, Phys. Lett. B 524 (2002) 400, Erratum.
- [20] G.W. Semenoff, K. Zarembo, Nucl. Phys. B 616 (2001) 34, hep-th/0106015.
- [21] G.W. Semenoff, K. Zarembo, Nucl. Phys. B (Proc. Suppl.) 108 (2002) 106, hep-th/0202156.
- [22] M. Bianchi, M.B. Green, S. Kovacs, JHEP 0204 (2002) 040, hep-th/0202003.
- [23] N. Drukker, hep-th/0605151.
- [24] J.M. Maldacena, Phys. Rev. Lett. 80 (1998) 4859, hep-th/9803002.

- [25] K. Zarembo, Phys. Rev. D 66 (2002) 105021, hep-th/0209095.
- [26] V. Pestun, K. Zarembo, Phys. Rev. D 67 (2003) 086007, hep-th/0212296.
- [27] A. Miwa, JHEP 0506 (2005) 050, hep-th/0504039.
- [28] N. Drukker, B. Fiol, JHEP 0601 (2006) 056, hep-th/0506058.
- [29] A. Miwa, T. Yoneya, hep-th/0609007.
- [30] D. Berenstein, R. Corrado, W. Fischler, J.M. Maldacena, Phys. Rev. D 59 (1999) 105023, hep-th/9809188.
- [31] K. Zarembo, Nucl. Phys. B 643 (2002) 157, hep-th/0205160.
- [32] H.J. Kim, L.J. Romans, P. van Nieuwenhuizen, Phys. Rev. D 32 (1985) 389.
- [33] S.M. Lee, S. Minwalla, M. Rangamani, N. Seiberg, Adv. Theor. Math. Phys. 2 (1998) 697, hep-th/9806074.
- [34] G.W. Semenoff, D. Young, Int. J. Mod. Phys. A 20 (2005) 2833, hep-th/0405288.
- [35] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, fifth ed., Academic Press, Boston, 1994, p. 962.
- [36] G.N. Watson, A Treatise on the Theory of Bessel Functions, second ed., Cambridge Univ. Press, London, 1966, p. 201.